Inverse Scattering Transform and Nonlinear Evolution Equations

Mark J. Ablowitz

Department of Applied Mathematics
University of Colorado, Boulder
October 2015

Outline

- I. Introduction, background
- II. Compatible linear systems, Lax pairs: $1 + 1$d
  - II.a Method (AKNS) to find linear compatible pairs: 2×2 systems, associated with nonlinear evolution eq – often with suitable symmetry–in physcially significant cases

    Examples: Korteweg-deVries (KdV), NLS (nonlinear Schrödinger), mKdV (modifed KdV), sine-Gordon (SG), ...
  - II.b New symmetry: integrable nonlocal NLS (2013)
  - II.c Compatibility Schrödinger scattering problem
  - II.d Classes of NL Evolution equations solvable by IST
  - II.e Remarks on $N \times N$ systems and extensions to $2 + 1$d
  - II.f Remarks on: compatible systems for discrete eq

Outline–con't

- III. Inverse Scattering Transform (IST): KdV

  Motivation: Fourier transforms and solution of linear PDEs

  KdV is related to linear Schrödinger scattering problem

  - IIIa. Direct scattering–analytic eigenfunctions, scattering data
  - IIIb. Inverse scattering: Riemann-Hilbert (RH) problems
  - IIIc. Time dependence of scattering data
  - IIId. Summary: Solution of KdV by IST
  - IIIe. Pure Solitons -'reflectionless potentials'
  - IIIf. Conserved quantities
  - IIIg. Inverse Problem: Connection to Gel'fand-Levitan-Marchenko (GLM) eq

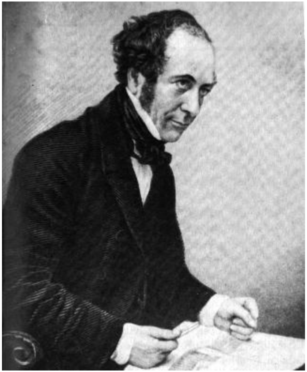
Outline–con't

- IV. Inverse Scattering Transform (IST): NLS, mKdV, SG,... These eq are related to 2x2 scattering problem with two potentials: q, r
  - IVa. Direct scattering–analytic eigenfunctions, scattering data, symmetry
  - IVb. Inverse scattering: Riemann-Hilbert problems
  - IVc. Time dependence of scattering data
  - IVd. Symmetry and IST solution of: NLS, mKdV, SG

    New symmetry — nonlocal NLS eq
  - IVe. Pure Solitons -'reflectionless potentials'
  - IVf. Conserved quantities
  - IVg. Inverse Problem: Connection to Gel'fand-Levitan-Marchenko eq
- Additional remarks and conclusions

I. Introduction–Background

- 1837–British Association for the Advancement of Science (BAAS) sets up a "Committee on Waves"; one of two members was J. S. Russell (Naval Scientist).
- 1837, 1840, 1844 (Russell's major effort): "Report on Waves" to the BAAS–describes a remarkable discovery



Russell-Wave of Translation

- Russell observed a localized wave: "rounded smooth...well-defined heap of water"
- Called it the "Great Wave of Translation" – later known as the solitary wave
- " Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon..."

Russell: to Mathematicians, Airy

Russell: "... it now remained for the mathematician to predict the discovery after it had happened..."
Leading British fluid dynamics researchers doubted the importance of Russell's solitary wave. G. Airy (below): wave was linear

Stokes

1847–G. Stokes : Stokes worked with nonlinear water wave equations and found a traveling periodic wave where the speed depends on amplitude (ambivalent w/r Russell). Stokes made many other critical contributions to fluid dynamics –"Navier-Stokes equations"

Boussinesq, Korteweg-deVries

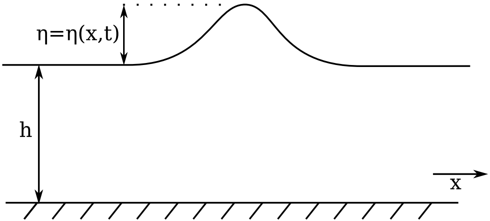
- 1871-77 – J. Boussinesq (left): new nonlinear eqs. and solitary wave solution for shallow water waves
- 1895 –D. Korteweg (right) & G. deVries: also shallow water waves ("KdV" eq.); NL periodic sol'n: "cnoidal" wave; limit case: the solitary wave (also see E. deJager '06: comparison Boussinesq – KdV)
- Russell's work was (finally) confirmed


KdV Equation –1895

KdV eq –1895

$$\frac{1}{\sqrt{gh}}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \frac{h^2}{2}(\frac{1}{3} - \hat{T})\eta_{xxx} = 0$$

where $\eta(x,t)$ is wave elevation above mean height h; g is gravity and $\hat{T}$ is normalized surface tension ($\hat{T} = \frac{T}{\rho g h^2}$)



KdV Eq.–con't

- nondimensional KdV eq.

$$u_t + 6uu_x + u_{xxx} = 0$$

- solitary wave:

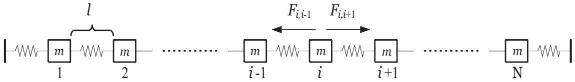
$$u = 2\kappa^2 \mathrm{sech}^2 \kappa(x - 4\kappa^2 t - x_0), \;\; \kappa, x_0 \; const$$

Solitary wave video

Click for solitary wave video

KdV –Modern Times

- 1895-1960 – Korteweg & deVries (KdV): water waves...
- 1960's – mathematicians developed approx methods to find reduced eq governing physical systems; KdV is an important "universal" eq
- 1960s M. Kruskal: 'FPU' (Fermi-Pasta-Ulam, 1955) problem



with force law: $F(\Delta) = -k(\Delta + \alpha\, \Delta^2), \alpha$ const; M.K. finds KdV eq in the continuum limit

KdV –Modern Times–con't

- 1965 –computation on KdV eq.

$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

N. Zabusky, M. Kruskal introduced the term *Solitons*

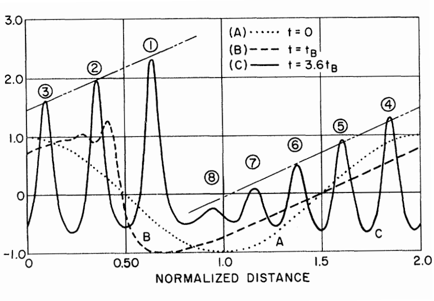


Figure : Calculations of the KdV Eq. with $\delta^2 \approx 0.02$ — from numerical calculations of ZK 1965

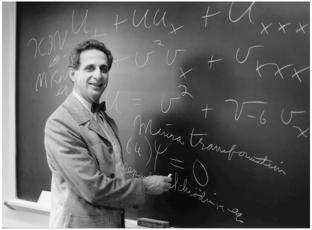
KdV –Modern Times–con't

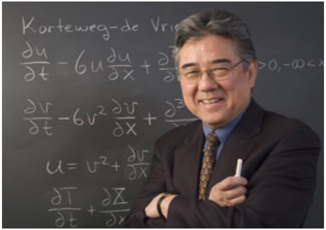
Kruskal and Miura study cons laws of KdV eq & modified KdV (mKdV) eq. Below KdV eq. left; mKdV eq right:

$$u_t + 6uu_x + u_{xxx} = 0, \quad v_t - 6v^2 v_x + v_{xxx} = 0$$

Miura finds a transformation between KdV and mKdV:

$$u = -(v_x + v^2)$$





KdV leads the way to IST

- Miura Transf leads to scattering problem and linearization of KdV: $v = \phi_x / \phi$

$$\phi_{xx} + (k^2 + u(x,t))\phi = 0, \quad \phi_t = M\phi$$

  k constant

- 1967 – Method to find solution of KdV: Gardner, Greene, Kruskal, Miura
- 1970's-present – KdV developments led to new methods & results in math physics
- Termed Inverse Scattering Transform (IST)–find solitons as special solutions

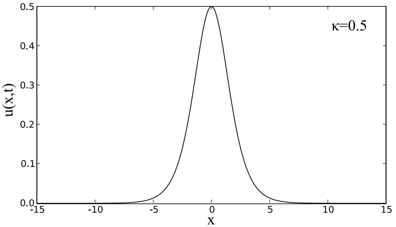
KdV Solitary Wave -Soliton

Normalized equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

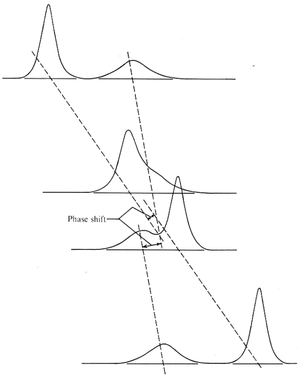
Soliton: $u_s(x,t) = 2\kappa^2 \operatorname{sech}^2 \kappa(x - 4\kappa^2 t - x_0)$

One eigenvalue: $u_{max} = 2\kappa^2$; speed $= 2u_{max}$, $x_0 = 0$



KdV –Two Soliton Interaction

KdV eq. with two eigenvalues: two solitons



Solitons: speed and amplitude preserved upon interaction

NLS is Integrable

Another important integrable eq. is the nonlinear Schrödinger eq. (NLS; Zakharov, Shabat, 1971)

$$iq_t = q_{xx} + Vq; \;\; V = \pm 2qq^*(x,t), \;\; * = cc$$

Related to

$$\phi_x = \begin{pmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{pmatrix} \phi \;\; \text{with} \;\; r(x,t) = \mp q^*(x,t)$$

$$\phi_t = M\phi, \qquad M = M[q,r], 2x2$$

k is constant

'Nonlocal NLS' is Integrable

A 'nonlocal NLS' eq is integrable:

$$iq_t = q_{xx} + Vq; \quad V = \pm 2q(x,t)q^*(-x,t)$$

Nonlocal NLS is related to

$$\phi_x = \begin{pmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{pmatrix} \phi \quad \text{with} \quad r(x,t) = \mp q^*(-x,t)$$

k is constant; MJA, Z. Musslimani, 2013

II. Compatible linear systems, Lax Pairs $1 + 1$d

Lax (1968) considered two operators; i.e. operator 'pair'– in general:

$$\mathcal{L}v = \lambda v$$
$$v_t = \mathcal{M}v$$

For KdV

$$\mathcal{L} = \partial_x^2 + u$$
$$\mathcal{M} = u_x + \gamma - (2u + 4\lambda)\partial_x = \gamma - 3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}$$

where γ is const and λ is a spectral parameter with $\lambda_t = 0$ 'isospectral flow'

Lax Pairs –con't

Take $\partial/\partial t$ of $\mathcal{L}v = \lambda v$:

$$\mathcal{L}_t v + \mathcal{L}v_t = \lambda_t v + \lambda v_t;$$

Use $v_t = \mathcal{M}v$

$$\begin{aligned} \mathcal{L}_t v + \mathcal{L}\mathcal{M}v &= \lambda_t v + \lambda\mathcal{M}v = \lambda_t v + \mathcal{M}\lambda v \\ &= \lambda_t v + \mathcal{M}\mathcal{L}v \quad => \end{aligned}$$

$$[\mathcal{L}_t + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})]\, v = \lambda_t v$$

Hence to find nontrivial ef $v(x,t)$

$$\mathcal{L}_t + [\mathcal{L},\mathcal{M}] = 0 \quad (La) \quad \text{where} \quad [\mathcal{L},\mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$$

if and only if $\lambda_t = 0$; (La) called Lax eq

Compatible Matrix Systems

Extension:

$$v_x = \mathbf{X}v, \qquad v_t = \mathbf{T}v;$$

where v is an n-d vector and $\mathbf{X}$ and $\mathbf{T}$ are $n \times n$ matrices : $\mathbf{X} = \mathbf{X}[\mathbf{u}; \lambda], \mathbf{T} = \mathbf{T}[\mathbf{u}; \lambda]$

Require compatibility: $v_{xt} = v_{tx}$, then

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = 0$$

and require e-value dependence to be *isospectral*. Above eq more general than Lax pair: allows more gen'l e-value dependence than $\mathcal{L}v = \lambda v$

2×2 Matrix Systems

Soon after KdV developments and Lax' ideas, Zakarov-Shabat (1971) found compatible pair and method of sol'n of NLS. AKNS (1973) generalized this to class of eq including NLS, mKdV, SG etc with following.

E-value prob (RHS: $\mathbf{X}$):

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= ikv_2 + r(x,t)v_1 \end{aligned}$$

Time dependence (RHS: $\mathbf{T}$)

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2 \\ v_{2,t} &= Cv_1 + Dv_2 \end{aligned}$$

where A, B, C and D functionals of $q(x,t)$, $r(x,t)$ and k

2×2 Matrix Systems–Special Cases

Note when when $r(x,t) = -1$, then from

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= \quad ikv_2 + r(x,t)v_1 = ikv_2 - v_1 \end{aligned}$$

we can solve for v_1 in terms of v_2; find v_2 satisfies:

$$v_{2,xx} + (k^2 + q)v_2 = 0$$

i.e the time independent Schrödinger e-value prob–which is related to KdV

Method below yields physically interesting NL evolution eq when $r = -1$, $r = \mp q^*$, $r = \mp q$, q real

2×2 Matrix Systems–con't

Consider the 2×2 compatible matrix system

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= \quad ikv_2 + r(x,t)v_1 \end{aligned}$$

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2 \\ v_{2,t} &= Cv_1 + Dv_2 \end{aligned}$$

Namely require $v_{j,xt} = v_{j,tx}, \;\; j = 1,2$, and $dk/dt = 0$: isospectral flow

This yields two eq of form: $\Gamma_j^1 v_1 + \Gamma_j^2 v_2 = 0, \;\; j = 1,2;$ we take $\Gamma_j^1 = \Gamma_j^2 = 0$

2×2 Matrix Systems–con't

This leads to $D = -A$ and three eq for A, B, C

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2ikB &= q_t - 2Aq \\ C_x - 2ikC &= r_t + 2Ar \end{aligned}$$

Note the e-value dependence k in coef of B, C 2nd 3rd eq
Look for sol'ns A, B, C in finite powers of k

$$A = \sum_{j=0}^{n} A_j k^j, \qquad B = \sum_{j=0}^{n} B_j k^j, \qquad C = \sum_{j=0}^{n} C_j k^j$$

Substitution yields eq which determine A_j, B_j, C_j and leave two additional constraints: NL evolution eq

2×2 Matrix Systems–Example

$$
\begin{aligned}
A_x &= qC - rB \\
B_x + 2ikB &= q_t - 2Aq \\
C_x - 2ikC &= r_t + 2Ar
\end{aligned}
$$

Example: $n = 2,\ A = A_2 k^2 + A_1 k + A_0$ etc.
The coefficients of k^3 give $B_2 = C_2 = 0$; at order k^2, we obtain $A_2 = a =$ const etc.
Find after some algebra: coupled NL evoln eq (constarint on sol'ns of A, B, C eq)

$$
\begin{aligned}
-\frac{1}{2} a q_{xx} &= q_t - a q^2 r \\
\frac{1}{2} a r_{xx} &= r_t + a q r^2
\end{aligned}
$$

2×2 Matrix Systems–NLS

If $r = \mp q^*$ and $a = 2i$, then find:

$$iq_t = q_{xx} \pm 2q^2 q^* \quad \text{NLS}$$

Both focusing $(+)$ and defocusing $(-)$ cases inlcuded
Summary $n = 2$ with $r = \mp q^*$ find

$$\begin{aligned} A &= 2ik^2 \mp iqq^* \\ B &= 2qk + iq_x \\ C &= \pm 2q^* k \mp iq_x^* \end{aligned}$$

provided that $q(x,t)$ satisfies the NLS eq
and recall: $dk/dt = 0$: isospectral flow

2×2 Matrix Systems–con't

$n = 3$, $A = A_3 k^3 + A_2 k^2 + A_1 k + A_0$ etc, find:

$$A = a_3 k^3 + a_2 k^2 + \frac{1}{2}(a_3 qr + a_1)k + \frac{a_2}{2} qr - \frac{ia_3}{4}(qr_x - rq_x) + a_0$$
$$B = ia_3 qk^2 + \left(ia_2 q - \frac{a-3}{2} q_x\right) k + \left[ia_1 q - \frac{a_2}{2} q_x + \frac{ia_3}{4}(2q^2 r - q_{xx})\right]$$
$$C = ia_3 rk^2 + \left(ia_2 r + \frac{a_3}{2} r_x\right) k + \left[ia_1 r + \frac{a_2}{2} r_x + \frac{ia_3}{4}(2r^2 q - r_{xx})\right]$$

$a_j, j = 0, 1, 2, 3$ are arb const. with 2 NL evoln eq (constraints)

$$q_t + \frac{ia_3}{4}(q_{xxx} - 6qrq_x) + \frac{a_2}{2}(q_{xx} - 2q^2 r) - ia_1 q_x - 2a_0 q = 0$$
$$r_t + \frac{ia_3}{4}(r_{xxx} - 6qrr_x) - \frac{a_2}{2}(r_{xx} - 2qr^2) - ia_1 r_x + 2a_0 r = 0$$

2×2 –KdV,mKdV

With $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = -1$, obtain the KdV eq:

$$q_t + 6qq_x + q_{xxx} = 0$$

If $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$ and $r = \mp q$, real, obtain the mKdV eq

$$q_t \pm 6q^2 q_x + q_{xxx} = 0$$

Have already seen: if $a_0 = a_1 = a_3 = 0$, $a_2 = -2i$ and $r = \mp q^*$, then we obtain the NLS eq

$$iq_t = q_{xx} \pm 2q^2 q^*$$

2×2 –Sine-Gordon, Sinh-Gordon Eq

Another ex. $n = -1$; take:

$A = \frac{a(x,t)}{k}, \qquad B = \frac{b(x,t)}{k}, \qquad C = \frac{c(x,t)}{k}$

Find eq for a, b, c; special cases are

$$(i): \quad a = \frac{i}{4}\cos u, \qquad b = -c = \frac{i}{4}\sin u, \qquad q = -r = -\frac{1}{2}u_x$$

and u satisfies the Sine–Gordon eq:

$$u_{xt} = \sin u$$

(ii): $\quad a = \frac{i}{4}\cosh u, \qquad b = -c = -\frac{i}{4}\sinh u, \qquad q = r = \frac{1}{2}u_x$

and u satisfies the Sinh–Gordon eq

$$u_{xt} = \sinh u$$

2×2– New Symmetry

If $r(x,t) = \mp q^*(-x,t)$ then for quadratic expansion in k find

$$iq_t = q_{xx} \pm 2q^2(x,t)q^*(-x,t) \qquad \text{Nonlocal NLS}$$

or written as

$$iq_t = q_{xx} \pm V[q]q(x,t) \qquad V[q] = q(x,t)q^*(-x,t)$$

Schrödinger Eigenvalue Problem

Originally KdV eq was related to the time independent Schrödinger e-value prob

Same method that works for 2×2 problem (when $r = -1$) also can be used directly
Compatible system:

$$v_{xx} + (\lambda + q)v = 0$$

$$v_t = Av + Bv_x$$

Compatibility: $(v_{xx})_t = (v_t)_{xx}$ yields eq for A, B (coef of v and v_x):

$$A_{xx} - 2B_x(\lambda + q) - Bq_x + q_t = 0$$

$$B_{xx} + 2A_x = 0$$

Schrödinger Eigenvalue Problem–con't

To find A, B let:

$$A = \sum_{j=0}^{n} A_j \lambda^j, \qquad B = \sum_{j=0}^{n} B_j \lambda^j$$

Substituting above into A, B eq and equating powers of λ yields A_j, B_j, j=1,2...n, and a constraint which is the NL evol eq.

Ex. $n = 1$ if take: $A_1 = 0, A_0 = q_x$, $B_1 = 4, B_0 = -2q$ find KdV eq.

$$q_t + 6qq_x + q_{xxx} = 0$$

2×2–General Class of NL Eq

A, B, C eq are linear eq that be solved for decaying q, r subject to constraint; find:

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + 2A_\infty(L)\begin{pmatrix} r \\ q \end{pmatrix} = 0$$

where $A_\infty(k) = \lim_{|x|\to\infty} A(x,t,k)$; $A_\infty(k)$ can be the ratio of two entire functions; L is

$$L = \frac{1}{2i}\begin{pmatrix} \partial_x - 2r(I_- q) & 2r(I_- r) \\ -2q(I_- q) & -\partial_x + 2q(I_- r) \end{pmatrix}$$

where $\partial_x \equiv \partial/\partial x$ and $(I_- f)(x) \equiv \int_{-\infty}^{x} f(y)dy$

2×2–General Class of NL Eq–con't

Ex. $A_\infty(k) = 2ik^2$ find:

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t = -4iL^2 \begin{pmatrix} r \\ q \end{pmatrix} = -2L \begin{pmatrix} r_x \\ q_x \end{pmatrix} = i \begin{pmatrix} r_{xx} - 2r^2 q \\ q_{xx} - 2q^2 r \end{pmatrix}$$

With $r = \mp q^*$ we have the NLS eq

$$iq_t = q_{xx} \pm 2q^2 q^* \qquad \text{NLS}$$

$A_\infty(k)$ can be related to the linear dispersion relation of constraint eq; i.e. if $q(x,t) = exp(i(kx - \omega_q(2k)t))$ we find that

$$A_\infty(k) = -\frac{i}{2}\omega_q(2k)$$

For NLS $\omega_q(k) = -k^2$ so $A_\infty(k) = 2ik^2$

Other Eigenvalue Problems

There have been numerous applications and generalizations of these method. For example the matrix generalization of 2×2 system; to $N \times N$ systems i.e.

$$\frac{\partial v}{\partial x} = ik\mathbf{J}\mathbf{v} + \mathbf{Q}\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = \mathbf{T}\mathbf{v}$$

where $\mathbf{Q}$ are $N \times N$ matrices with $Q^{ii} = 0$,
$\mathbf{J} = \mathrm{diag}(J^1, J^2, \ldots, J^N)$, with $J^i \neq J^j$ for $i \neq j$ and $\mathbf{v}(x, t)$ is an N-dimensional vector
$\mathbf{T}$ is also an $N \times N$ matrix and can be expanded in powers of k

Find numerous interesting compatible NL evol eq such as N wave eq, Boussinesq eq etc.

$2 + 1$d 'scattering' Problems

There are compatible systems in $2 + 1$d and discrete systems
In $2 + 1$d perhaps the best known is the $N \times N$ linear system:

$$\frac{\partial v}{\partial x} = \mathbf{J}\frac{\partial v}{\partial y} + \mathbf{Q}\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = \mathbf{T}\mathbf{v}$$

Compatible systems are obtained by expanding $\mathbf{T}$ in powers of $\frac{\partial}{\partial y}$
Find N wave, Davey-Stewartson (2×2 system with $r = \mp q^*$), and Kadomstsev-Petviashvili (KP) eq (2×2 system with $r = -1$):

$$(q_t + 6qq_x + q_{xxx})_x + \sigma^2 q_{yy} = 0 \quad \text{KP}$$

where $\sigma^2 = \mp 1$: so called KP I,II eq

In scalar form spatial 'scattering' eq is $\sigma v_y + v_{xx} + uv = 0$

Discrete Eigenvalue Problems

Recall the continuous 2×2 system

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= \quad ikv_2 + r(x,t)v_1 \end{aligned}$$

Discretizing $v_{j,x} \approx \frac{v_{j,n+1} - v_{j,n}}{h}$ and calling $z = e^{ikh} \approx 1 + ikh + \cdots$ and $Q_n(t) = hq_n, R_n(t) = hr_n$ etc leads to the following discrete 2×2 eigenvalue problem

$$\begin{aligned} v_{1,n+1} &= zv_{1,n} + Q_n(t)v_{2,n} \\ v_{2,n+1} &= \frac{1}{z}v_{2,n} + R_n(t)v_{1,n} \end{aligned}$$

Discrete Eigenvalue Problems–con't

To

$$
\begin{aligned}
v_{1,n+1} &= z v_{1,n} + Q_n(t) v_{2,n} \\
v_{2,n+1} &= \frac{1}{z} v_{2,n} + R_n(t) v_{1,n}
\end{aligned}
$$

we add time dependence

$$
\begin{aligned}
\frac{dv_{1,n}}{dt} &= A v_{1,n} + B v_{2,n} \\
\frac{dv_{2,n}}{dt} &= C v_{1,n} + D v_{2,n}
\end{aligned}
$$

Making these two eq compatible and expanding A_n, B_n, C_n, D_n in finite Laurent series in z yields NL Evol eq as constraints

Discrete Eigenvalue Problems–con't

Ex. Expanding

$A_n = \sum_{j=-2}^{2} A_{j,n} z^j$ similar for B_n, C_n, D_n eventually yields

$$\begin{aligned} i\frac{d}{dt}Q_n &= Q_{n+1} - 2Q_n + Q_{n-1} - Q_n R_n \left(Q_{n+1} + Q_{n-1}\right) \\ -i\frac{d}{dt}R_n &= R_{n+1} - 2R_n + R_{n-1} - Q_n R_n \left(R_{n+1} + R_{n-1}\right) \end{aligned}$$

With $R_n = \mp Q_n^*$ we have the integrable discrete NLS eq

$$i\frac{d}{dt}Q_n = Q_{n+1} - 2Q_n + Q_{n-1} - |Q_n|^2 \left(Q_{n+1} + Q_{n-1}\right)$$

or with $Q_n(t) = hq_n(t)$

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) \pm |q_n|^2 \left(q_{n+1} + q_{n-1}\right)$$

III. Inverse Scattering Transform (IST) for KdV

Motivation: linear Fourier Transform (FT)

Consider the linear evol eq

$$u_t = \sum_{j=0}^{N} a_j \partial_x^j u, \quad a_j \in \mathbb{R} \ \text{ const}$$

The soln $u(x,t)$ can be found via FT as

$$u(x,t) = \frac{1}{2\pi} \int b(k,t) e^{ikx} dk \qquad \text{(FT)}$$

where it is assumed that u is smooth and $|u| \to 0$ as $|x| \to \infty$ sufficiently rapidly; unless otherwise specified: $\int = \int_{-\infty}^{\infty}$

FourierTransform–con't

Substituting FT into linear eq yields (assume interchanges etc)

$$\int \mathrm{e}^{ikx}\{b_t - b\sum_{j=0}^{N}(ik)^j a_j\}\, dk = 0 \quad \text{or} \quad b_t = b\sum_{j=0}^{N}(ik)^j a_j$$

So

$$b(k,t) = b_0(k)\mathrm{e}^{-i\omega(k)t}, \quad \omega(k) = i\sum_{j=0}^{N}(ik)^j a_j$$

Typically when $\omega(k) \in \mathbb{R}$ $(a_{2j} = 0, j = 0, 1...)$, it is called the dispersion relation. Thus the soln is given by

$$u(x,t) = \frac{1}{2\pi}\int b_0(k)e^{i[kx-\omega(k)t]}\, dk$$

For $u(x,t) \in \mathbb{R}$ require symmetry: $b_0^*(-k) = b_0(k)$

FourierTransform–Linear KdV

The previous result shows that for the linear KdV eq

$$u_t + u_{xxx} = 0$$

from $u = e^{i[kx-\omega(k)t]}$ the linear dispersion relation is: $\omega = -k^3$ and the FT soln is given by

$$u(x,t) = \frac{1}{2\pi}\int b_0(k)e^{i[kx+k^3t]}\,dk$$

The soln process via FT is given by

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct FT}} & b(k,0) = b_0(k) \\ & & \Big\downarrow t:\ \text{time evolution} \\ u(x,t) & \xleftarrow{\text{inverse FT}} & b(k,t) = b_0(k)e^{-i\omega(k)t} \end{array}$$

IST for KdV

Compatibility of the following system

$L: \; v_{xx} + (\lambda + u(x,t))v = 0$ and $M: \; v_t = (\gamma + u_x)v + (4\lambda - 2u)v_x$

where γ=const and $\lambda_t = 0$ yields the KdV eq

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{KdV}$$

Soln process via IST:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{Direct Scattering}} & L: S(k,0) \\ & & \Big\downarrow \; t\text{: time evolution: M} \\ u(x,t) & \xleftarrow{\text{Inverse Scattering}} & S(k,t) \end{array}$$

Direct Scattering–con't

Begin with discussion of direct scattering problem. Let $\lambda = k^2$, then L (scattering) operator is:

$$L: \quad v_{xx} + \left(u(x) + k^2\right) v = 0$$

note suppression the time dependence in u. Assume that $u(x) \in \mathbb{R}$ and decays sufficiently rapidly, e.g. u lies in the space of functions

$$L_n^1: \quad \int_{-\infty}^{\infty} (1 + |x|^n)|u(x)|dx < \infty, \quad n \geq 2$$

Associated with operator L are 2 sets of efcns for real k that are bounded for all values of x, and that have appropriate analytic extensions into UHP–k, LHP–k

Direct Scattering–con't

Appropriate efcns associated with operator L are defined from their BCs; i.e. identify 4 efcns defined by the following asymptotic BCs

$$\begin{aligned} \phi(x;k) &\sim e^{-ikx}, \quad \bar{\phi}(x;k) \sim e^{ikx} \quad \text{as} \quad x \to -\infty \\ \psi(x;k) &\sim e^{ikx}, \quad \bar{\psi}(x;k) \sim e^{-ikx} \quad \text{as} \quad x \to \infty \end{aligned}$$

So, e.g. $\phi(x,k)$ is a soln of L eq which tends to e^{-ikx} as $x \to -\infty$ etc. Note: $\bar{\phi}$ does not represent cc; rather *=cc
From L and BCs and $u(x) \in \mathbb{R}$ have symmetries:

$$\phi(x;k) = \bar{\phi}(x;-k) = \phi^*(x,-k)$$

$$\psi(x;k) = \psi(x;-k) = \psi^*(x,-k)$$

Direct Scattering–con't

The *Wronskian* of 2 solns ψ, ϕ is defined as

$$W(\phi, \psi) = \phi\psi_x - \phi_x\psi$$

and from Abel's Theorem, the Wronskian is const.
Hence from $\pm\infty$:

$$W(\psi, \bar{\psi}) = -2ik = -W(\phi, \bar{\phi})$$

Since L is a linear 2nd order ODE, from linear independence of its solutions we obtain the following completeness relationships between the efcns

$$\begin{aligned} \phi(x;k) &= \quad a(k)\bar{\psi}(x;k) + b(k)\psi(x;k) \\ \bar{\phi}(x;k) &= -\bar{a}(k)\psi(x;k) + \bar{b}(k)\bar{\psi}(x;k) \end{aligned}$$

For $u(x) \in \mathbb{R}$ only need first eq

Direct Scattering–con't

$a(k), b(k)$ can be expressed in terms of Wronskians:

$$a(k) = \frac{W(\phi(x;k), \psi(x;k))}{2ik}, \qquad b(k) = -\frac{W(\phi(x;k), \bar{\psi}(x;k))}{2ik}$$

Thus $\phi, \psi, \bar{\psi}$ determine $a(k), b(k)$ which are part of the 'scattering data'
Also have symmetries: $a(-k) = a^*(k); b(-k) = b^*(k)$ and unitarity:

$$|a(k)|^2 - |b(k)|^2 = 1, \quad k \in \mathbb{R}$$

Direct Scattering–con't

It is more convenient to work with modified efcns $M(x;k), N(x;k), \bar{N}(x;k)$:

$$M(x;k) = \phi(x;k)e^{ikx}$$
$$N(x;k) = \psi(x;k)e^{ikx}, \qquad \bar{N}(x;k) = \bar{\psi}(x;k)e^{ikx}$$

Completeness of efcns implies

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)N(x;k)$$

where $\quad \rho(k) = \frac{b(k)}{a(k)}$

$\tau(k) = 1/a(k)$ and $\rho(k)$ are called the **transmission** and **reflection** coefs

Direct Scattering–con't

$\psi(x;k) = \bar{\psi}(x;-k)$ implies $N(x;k) = \bar{N}(x;-k)e^{2ikx}$

Due to this symmetry will only need 2 efcns. Namely, from completeness:

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

where $\rho(k) = \frac{b(k)}{a(k)}$

(*) will be a fundamental eq. Later will show that (*) leads to a generalized **Riemann-Hilbert boundary value problem** (RH)

Analyticity of Efcns

Theorem

For $u \in L^1_2 : \int_{-\infty}^{\infty}(1+|x|^2)|u| < \infty$

(i) $M(x;k)$ *and* $a(k)$ *are analytic fcns of* k *for* $Im k > 0$ *and tend to unity as* $|k| \to \infty$*; they are continuous on* $Im k = 0$*;*

(ii) $\bar{N}(x;k)$ *and* $\bar{a}(k)$ *are analytic fcns of* k *for* $Im k < 0$ *and tend to unity as* $|k| \to \infty$*; they are continuous on* $Im k = 0$ *Moreover, the solutions of the corresponding integral equations are unique.*

Using Green's fcn techniques may show that $M(x;k), \bar{N}(x;k)$ satisfy the following Volterra integral eq

$$
\begin{aligned}
M(x;k) &= 1 + \frac{1}{2ik}\int_{-\infty}^{x}\left\{1 - e^{2ik(x-\xi)}\right\} u(\xi) M(\xi;k) d\xi \\
\bar{N}(x;k) &= 1 - \frac{1}{2ik}\int_{x}^{\infty}\left\{1 - e^{-2ik(\xi-x)}\right\} u(\xi) \bar{N}(\xi;k) d\xi
\end{aligned}
$$

Proof: Convergence of Neumann series

Potential and Efcns

From efcn can determine potential u

Using

$$\bar{N}(x;k) = 1 - \frac{1}{2ik}\int_x^\infty \left\{1 - e^{-2ik(\xi - x)}\right\} u(\xi)\bar{N}(\xi;k)d\xi$$

then for $\mathrm{Im}k \geq 0$, as $k \to \infty$, iteration and Reimann-Lesbegue Lemma implies:

$$\bar{N}(x;k) \sim 1 - \frac{1}{2ik}\int_x^\infty u(\xi)d\xi \qquad (**)$$

Analyticity, RH Problem and Scattering Data

We will work with

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

where $\rho(k) = \frac{b(k)}{a(k)}$

Note: LHS: $\frac{M(x;k)}{a(k)}$ is analytic UHP-k/[zero's of $a(k)$]; RHS: $\bar{N}(x;k)$ is analytic LHP-k;

We will consider remaining term as the 'jump' (change) in analyticity across Rek axis

Required Scattering Data

Scattering data that will be needed: $\rho(k)$ and information about zero's of $a(k)$

For real $u(x)$ from operator L can show:

$a(k)$ has a **finite number of simple zero's on img axis**:
$a(k_j) = 0$; $\{k_j = i\kappa_j\}, j = 1, ...J; \kappa_j > 0$;
Note also $a(k) \to 1$ as $k \to \infty$, analytic UHP-k; continuous $Imk = 0$

At every zero $k_j = i\kappa_j$ there are L^2 bound states:
$\phi_j = \phi(x, k_j), \psi_j = \psi(x, k_j)$ such that $\phi_j = b_j\psi_j => M_j = b_j N_j$;
for inverse problem we will need: $C_j = b_j/a'(k_j); j = 1, ...J$

Next: Inverse Problem

Recall scheme:

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & L : S(k,0) \\
 & & \Big\downarrow {\scriptstyle t\text{: time evolution: M}} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & S(k,t)
\end{array}
$$

Next consider Inverse problem at fixed time

Inverse Scattering–Projection Operators

Recall

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

(*) is fundamental eq.
Apart from poles at $a(k_j) = 0$, $\frac{M(x;k)}{a(k)}$ is anal UHP; and $\bar{N}(x;k)$ is anal in LHP. $(*)$ a generalized (RH) prob'; it leads to an integral eq for $N(x;k)$

Use projection operators
Consider the $\mathcal{P}^{\pm}$ projection operator defined by

$$(\mathcal{P}^{\pm}f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\zeta)d\zeta}{\zeta-(k\pm i0)} = \lim_{\varepsilon\downarrow 0}\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\zeta)d\zeta}{\zeta-(k\pm i\varepsilon)}\right\}$$

Projection Operators–con't

If $f(k) = f_\pm(k)$ is anal in the UHP/LHP-k and $f_\pm(k) \to 0$ as $|k| \to \infty$ (for Im $k \gtrless 0$), then from contour integration:

$$(\mathcal{P}^\pm f_\mp)(k) = 0$$
$$(\mathcal{P}^\pm f_\pm)(k) = \pm f_\pm(k),$$

To most easily explain ideas, 1st assume that there are no poles, that is $a(k) \neq 0$. Then operating on (*) with $\mathcal{P}^-$:

$$\mathcal{P}^-\left[\left(\frac{M(x;k)}{a(k)} - 1\right)\right] = \mathcal{P}^-\left[(\bar{N}(x;k) - 1) + \rho(k)e^{2ikx}\bar{N}(x;-k)\right]$$

From Proj: LHS=0 (since assumed no zero's of $a(k)$); and $\mathcal{P}^-\left[(\bar{N}(x;k) - 1)\right] = -(\bar{N}(x;k) - 1)$ implies

Inverse Problem: no poles

$$\bar{N}(x;k) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta - (k - i0)} \qquad \text{(E1)}$$

Symmetry: $N(x;k) = e^{2ikx}\bar{N}(x;-k)$ => an integral eq

$$N(x;k) = e^{2ikx}\left\{1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + k + i0}\right\}$$

Reconstruction of the potential u; As $k \to \infty$ (E1) implies

$$\bar{N}(x;k) \sim 1 - \frac{1}{2\pi ik}\int_{-\infty}^{\infty} \rho(\zeta)N(x;\zeta)d\zeta \qquad \text{(E2)}$$

From direct integral eq (**): $\bar{N}(x;k) \sim 1 - \frac{1}{2ik}\int_x^{\infty} u(\xi)d\xi$;
comparing (**) & (E2):

$$u(x) = -\frac{\partial}{\partial x}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \rho(\zeta)N(x;\zeta)d\zeta\right\}$$

Inverse Problem: Including Poles

For the case when $a(k)$ has zeros, one can extend the above result; suppose

$$a(k_j = i\kappa_j) = 0, \quad \kappa_j > 0, \quad j = 1, \cdots J$$

then call

$$N_j(x) = N(x; k_j)$$

Subtracting the pole contributions and carrying out similar calculations as before leads to

$$N(x; k) = e^{2ikx} \left\{ 1 - \sum_{j=1}^{J} \frac{C_j N_j(x)}{k + i\kappa_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho(\zeta) N(x; \zeta) d\zeta}{\zeta + k + i0} \right\}$$

Inverse Problem: Including Poles–con't

To complete the system, evaluate at $k = k_p = i\kappa_p$

$$N(x;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{J}\frac{C_j N_j(x)}{k + i\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + k + i0}\right\}$$

$$N_p(x) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J}\frac{C_j N_j(x)}{i(\kappa_p + \kappa_j)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + i\kappa_p}\right\}$$

for $p = 1, ...J$. Above is a coupled system of integral eq for $N(x,k); \{N_j(x) = N(x,k_j)\}, j = 1, \cdots, L$

From these eq $u(x)$ is reconstructed from

$$u(x) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\rho(\zeta)N(x;\zeta)d\zeta\right\}$$

IST – So Far

So far in the IST process direct and inverse problem have been discussed.

Direct problem (from operator L): $u(x) \to \mathcal{S}(k)$

Inverse problem: $\mathcal{S}(k) = \{\rho(k), \ \ \{\kappa_j, C_j\}\ \} \to u(x)$
Direct and inverse problems are the NL analogues of the direct and inverse Fourier transform

Next need time dependence; recall:

$$\begin{tikzcd} u(x,0) \arrow[r, "\text{Direct Scattering}"] & L : S(k,0) \arrow[d, "t\text{: time evolution: M}"] \\ u(x,t) & S(k,t) \arrow[l, "\text{Inverse Scattering}"'] \end{tikzcd}$$

IST: Time Dependence

For time dependence use associated time evolution operator: M which for the KdV eq is

$$v_t = Mv = (u_x + \gamma)v + (4k^2 - 2u)v_x$$

with γconst. With $v = \phi(x, k)$ and using

$$\phi(x, t; k) = M(x, t; k)e^{-ikx},$$

M then satisfies

$$M_t = (\gamma - 4ik^3 + u_x + 2iku)M + (4k^2 - 2u)M_x$$

Also recall

$$M(x, t; k) = a(k, t)\bar{N}(x, t; k) + b(k, t)N(x, t; k)$$

IST: Time Dependence

The asymptotic behavior of $M(x,t;k)$ is given by

$$M(x,t;k) \to 1, \qquad \text{as} \quad x \to -\infty$$
$$M(x,t;k) \to a(k,t) + b(k,t)e^{2ikx} \qquad \text{as} \quad x \to \infty$$

From

$$M_t = (\gamma - 4ik^3 + u_x + 2iku)M + (4k^2 - 2u)M_x$$

and using the fact that $u \to 0$ rapidly as $x \to \pm\infty$, find

$$\gamma - 4ik^3 = 0, \quad x \to -\infty$$

$$a_t + b_t e^{2ikx} = 8ik^3 b e^{2ikx}, \quad x \to +\infty$$

and by equating coef of e^0, e^{2ikx} find

$$a_t = 0, \qquad b_t = 8ik^3 b$$

IST: Time Dependence–con't

Solving a, b eq yields

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)\exp(8ik^3 t) \quad \text{so}$$

$$\rho(k,t) = \frac{b(k,t)}{a(k,t)} = \rho(k,0)e^{8ik^3 t}$$

$a(k_j) = 0$ implies zero's (evalues) k_j which are finite in number, simple, and lie on the Im axis, also satisfy

$$k_j = i\kappa_j = \text{constant}, \qquad j = 1, \dots, J$$

Since the evalues are const in time; so this is an "isospectral flow" Also find the time dependence of the $C_j(t)$ is given by

$$C_j(t) = C_j(0)e^{8ik_j^3 t} = C_j(0)e^{8\kappa_j^3 t} \qquad j = 1, \dots J$$

IST

Thus we have the time dependence scattering data:

$\mathcal{S}(k,t) = \{\rho(k,t), \ \ \{\kappa_j, C_j(t)\} \ \ j = 1, \ldots, J\}$; with

$\rho(k,t) = \rho(k,0)e^{8ik^3t}; \kappa_j = \text{const}; C_j(t) = C_j(0)e^{8\kappa_j^3 t} \ j = 1, \ldots J$

This completes the IST formulation:

$$\begin{tikzcd} u(x,0) \arrow[r, "\text{Direct Scattering}"] & L : S(k,0) \arrow[d, "t\text{: time evolution: M}"] \\ u(x,t) & S(k,t) \arrow[l, "\text{Inverse Scattering}"'] \end{tikzcd}$$

Inverse Problem: Including Poles–time included

To complete the system, evaluate at $k = k_p = i\kappa_p$

$$N(x,t;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{k+i\kappa_j} + \int_{-\infty}^{\infty} \frac{\rho(\zeta,t)N(x,t;\zeta)d\zeta}{2\pi i(\zeta+k+i0)}\right\}$$

$$N_p(x,t) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{i(\kappa_p+\kappa_j)} + \int_{-\infty}^{\infty} \frac{\rho(\zeta,t)N(x,t;\zeta)d\zeta}{2\pi i(\zeta+i\kappa_p)}\right\}$$

for $p = 1, ...J$. Above is a coupled system of integral eq for $N(x,k); \{N_j(x) = N(x,k_j)\}, j = 1, \cdots, L$
From these eq $u(x)$ is reconstructed from

$$u(x,t) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j(t)N_j(x,t) - \frac{1}{\pi}\int_{-\infty}^{\infty} \rho(\zeta,t)N(x,t;\zeta)d\zeta\right\}$$

'Pure' Solitons–Reflectionless Potls

'Pure' solitons are obtained by assuming $\rho(k,0) = 0$ 'reflectionless' potentials. From IST–need only the discrete contributions

$$N_p(x,t) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{i(\kappa_p + \kappa_j)}\right\}, \quad p = 1, \cdots, J$$

Above is a linear algebraic system for
$\{N_p(x,t) = N(x,t,k_p)\}, p = 1, .., J$
From these eq $u(x,t)$ is reconstructed from

$$u(x,t) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j(t)N_j(x,t)\right\}$$

IST–One Soliton

When there is only one ev ($J = 1$) find

$$N_1(x,t) - \frac{iC_1(0)}{2\kappa_1} e^{-2\kappa_1 x + 8\kappa_1^3 t} N_1(x,t) = e^{-2\kappa_1 x}$$

which yields $N_1(x,t)$ and $u(x,t)$:

$$N_1(x,t) = \frac{2\kappa_1 e^{-2\kappa_1 x}}{2\kappa_1 - iC_1(0) e^{-2\kappa_1 x + 8\kappa_1^3 t}}$$

$$u(x,t) = 2\frac{\partial}{\partial x}\left\{ e^{8\kappa_1^3 t} iC_1(0) N_1(x,t) \right\}$$

which leads to the familiar one soliton soln:

$$u(x,t) = 2\kappa_1^2 \operatorname{sech}^2\left\{ \kappa_1 (x - 4\kappa_1^2 t - x_1) \right\}$$

where x_1 is defined via $-iC_1(0) = 2\kappa_1 \exp(2\kappa_1 x_1)$

Conserved Quantities

May relate $a(k)$, which is a constant of motion, to an infinite number of conserved quantities from

$$a(k) = \frac{1}{2ik} W(\phi, \psi)$$
$$= \frac{1}{2ik}(\phi\psi_x - \phi_x\psi) = \lim_{x\to+\infty} \frac{1}{2ik}\left(\phi ik \mathrm{e}^{ikx} - \phi_x \mathrm{e}^{ikx}\right)$$

and developing large k expn for $\phi(x, t; k)$ as a functional of u
The first few nontrivial conserved quantities are found to be:

$$C_1 = \int_{-\infty}^{\infty} u dx, \quad C_3 = \int_{-\infty}^{\infty} u^2 dx, \quad C_5 = \int_{-\infty}^{\infty} (2u^3 - u_x^2) dx, \ldots$$

May use similar ideas to find conservation laws:

$$\partial_t T_j + \partial_x F_j = 0, \quad j = 1, 2 \ldots$$

IST–via Gel'fand-Levitan-Marchenko (GLM) Eq

The GLM eq may be derived from the RH formulation
$N(x,t;k)$ is written in terms of a triangular kernel:

$$N(x,t;k) = e^{2ikx}\left\{1 + \int_x^\infty K(x,s;t)e^{ik(s-x)}ds\right\}$$

Subst above into RH formulation and taking a FT yields

$$K(x,y;t)+F(x+y;t)+\int_x^\infty K(x,s;t)\,F(s+y;t)\,ds = 0, \qquad y > x$$

$$\text{where} \quad F(x;t) = \sum_{j=1}^{L}(-i)C_j(t)e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(k,t)e^{ikx}dk$$

and also find: $\qquad u(x,t) = 2\partial_x K(x,x;t)$
May get soliton solns from GLM; Rigorous inverse pb:
Deift-Trubowitz ('79); Marchenko ('86); ...

IV. IST: 2×2 Systems

Next study following 2×2 compatible systems:

$$v_x = Lv = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

$$v_t = Mv = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v$$

The 'scattering' eq may be written in the form:

$$v_x = (ik\mathbf{J} + \mathbf{Q})\, v \quad \text{where}$$

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

IST–2 × 2 Systems Direct Scattering

Recall: Soln process via IST:

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & L: S(k,0) \\
 & & \Big\downarrow {\scriptstyle t\text{: time evolution: M}} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & S(k,t)
\end{array}
$$

For

$$v_x = Lv = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

when $q, r \to 0$ sufficiently rapidly as $x \to \pm\infty$ the efcns are asymptotic to the solns of

$$v_x \sim \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix} v$$

Efcns–2×2 Systems

Key efcns defined by the following BCs:

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \qquad \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \qquad \text{as } x \to -\infty$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \qquad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \qquad \text{as } x \to +\infty$$

Convenient to work with efcns which have const BCs at infinity:

As $x \to -\infty$:

$$M(x,k) = e^{ikx}\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{M}(x,k) = e^{-ikx}\bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

As $x \to \infty$:

$$N(x,k) = e^{-ikx}\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{N}(x,k) = e^{ikx}\psi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Wronskian and Lin Indepence of Efcns

Let $\quad u(x,k) = \left(u^{(1)}(x,k), u^{(2)}(x,k)\right)^T \qquad$ and

$\qquad v(x,k) = \left(v^{(1)}(x,k), v^{(2)}(x,k)\right)^T \qquad$ be 2 solns of L eq

The Wronskian of u and v is

$$W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$$

which satisfies

$$\frac{d}{dx}W(u,v) = 0 => W(u,v) = W_0 \;\; \text{const}$$

From the asymptotic behavior of the efcns find:

$$\begin{aligned} W\left(\phi, \bar{\phi}\right) &= \lim_{x\to-\infty} W\left(\phi(x,k), \bar{\phi}(x,k)\right) = 1 \\ W\left(\psi, \bar{\psi}\right) &= \lim_{x\to+\infty} W\left(\psi(x,k), \bar{\psi}(x,k)\right) = -1 \end{aligned}$$

Thus the solns ϕ and $\bar{\phi}$ are linearly independent, as are ψ and $\bar{\psi}$

Efcns and Scattering Data

Completeness of efcns implies

$$\begin{aligned} \phi(x,k) &= b(k)\psi(x,k) + a(k)\bar{\psi}(x,k) \\ \bar{\phi}(x,k) &= \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k) \end{aligned}$$

It follows that $a(k), \bar{a}(k), b(k), \bar{b}(k)$ (scatt data) satisfy:

$$\begin{aligned} a(k) &= W(\phi,\psi), \qquad \bar{a}(k) = W(\bar{\psi},\bar{\phi}) \\ b(k) &= W(\bar{\psi},\phi), \qquad \bar{b}(k) = W\left(\bar{\phi},\psi\right) \end{aligned}$$

Also have unitarity:

$$a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1, \quad k \in \mathbb{R}$$

Efcns and Scattering Data–con't

In terms of $M, N, \bar{M}, \bar{N}$ completeness implies:

$$\begin{aligned} \frac{M(x,k)}{a(k)} &= \bar{N}(x,k) + \rho(k) e^{2ikx} N(x,k) \\ \frac{\bar{M}(x,k)}{\bar{a}(k)} &= N(x,k) + \bar{\rho}(k) e^{-2ikx} \bar{N}(x,k) \end{aligned}$$

where the *reflection coefficients* are

$$\rho(k) = b(k)/a(k), \qquad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k)$$

The above eqs will be considered as generalized Riemann-Hilbert (RH) pbs. Need analyticity–next

Efcns–2×2 Systems: Diff Eq

The fcns $M(x,k), \bar{N}(x,k)$ satisfy the following DE for $\chi(x,k)$:

$$\partial_x \chi(x,k) = ik\left(\mathbf{J}+\mathbf{I}\right)\chi(x,k) + \left(\mathbf{Q}\chi\right)(x,k)$$

while the fcns $\bar{M}(x,k), N(x,k)$ satisfy the DE for $\bar{\chi}(x,k)$:

$$\partial_x \bar{\chi}(x,k) = ik\left(\mathbf{J}-\mathbf{I}\right)\bar{\chi}(x,k) + \left(\mathbf{Q}\bar{\chi}\right)(x,k)$$

where

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

and $\mathbf{I}$ is the 2×2 identity matrix. Via Green's fcns methods we may convert DE to an Integral eq

Efcns–2 $\times$ 2 Systems: Integral Eq

Efcns can be written in terms of Volterra integral eq:

$$
\begin{aligned}
M(x,k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_{+}(x-x',k)\mathbf{Q}(x')M(x',k)dx' \\
N(x,k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{+\infty} \bar{\mathbf{G}}_{+}(x-x',k)\mathbf{Q}(x')N(x',k)dx' \\
\bar{M}(x,k) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{+\infty} \bar{\mathbf{G}}_{-}(x-x',k)\mathbf{Q}(x')\bar{M}(x',k)dx' \\
\bar{N}(x,k) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_{-}(x-x',k)\mathbf{Q}(x')\bar{N}(x',k)dx'
\end{aligned}
$$

with ($\theta(x)$ Heaviside fcn):

$$
\mathbf{G}_{\pm}(x,k) = \pm\theta(\pm x)\begin{pmatrix} 1 & 0 \\ 0 & e^{2ikx} \end{pmatrix}, \; \bar{\mathbf{G}}_{\pm}(x,k) = \mp\theta(\mp x)\begin{pmatrix} e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix}
$$

Analyticity of Efcns

Theorem

If $q, r \in L^1(\mathbb{R})$, then $\{M(x,k), N(x,k), a(k)\}$ are analytic functions of k for $\mathrm{Im} k > 0$ and continuous for $\mathrm{Im} k \geq 0$, while $\{\bar{M}(x,k), \bar{N}(x,k), \bar{a}(k)\}$ are analytic functions of k for $\mathrm{Im} k < 0$ and continuous for $\mathrm{Im} k \leq 0$. Moreover, the solutions of the corresponding integral equations are unique.

Proof: Convergence of Neumann series

Large k Behavior

From the integral equations can compute the asymptotic expn as $k \to \infty$ (in the proper half-plane) for the efcns; find

$$
\begin{aligned}
M(x,k) &= \begin{pmatrix} 1 - \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \\ -\frac{1}{2ik} r(x) \end{pmatrix} + O(1/k^2) \\
\bar{N}(x,k) &= \begin{pmatrix} 1 + \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \\ -\frac{1}{2ik} r(x) \end{pmatrix} + O(1/k^2) \\
N(x,k) &= \begin{pmatrix} \frac{1}{2ik} q(x) \\ 1 - \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \end{pmatrix} + O(1/k^2) \\
\bar{M}(x,k) &= \begin{pmatrix} \frac{1}{2ik} q(x) \\ 1 + \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \end{pmatrix} + O(1/k^2)
\end{aligned}
$$

and $\quad a(k) = 1 + O(\frac{1}{k})$ and $\quad \bar{a}(k) = 1 + O(\frac{1}{k})$ as $k \to \infty$

Required Scattering Data

Scattering data that will be needed–in general position: $\rho(k), \bar{\rho}(k)$ and information about zero's (evalues) of $a(k), \bar{a}(k)$

For general $q(x), r(x)$ **proper** evalues correspond to L^2 bound states; they are assumed simple and not on the real k axis

At: $a(k_j) = 0, k_j = \xi_j + i\eta_j, \eta_j > 0, \quad j = 1, 2, ..., J$ with

$$\phi_j(x) = b_j \psi_j(x) \text{ where } \phi_j(x) = \phi(x, k_j) \text{ etc}$$

This implies
Similarly at: $\bar{a}(\bar{k}_j) = 0, \bar{k}_j = \bar{\xi}_j - i\bar{\eta}_j, \ \bar{\eta}_j > 0, \quad j = 1, 2, ..., \bar{J}$ with

$$\bar{\phi}_j(x) = \bar{b}_j \bar{\psi}_j(x)$$

Required Scattering Data–con't

In terms of $M, N, \bar{M}, \bar{N}$ proper evalues correspond to

$$M_j(x) = b_j e^{2ik_jx} N_j(x)\,, \qquad \bar{M}_j(x) = \bar{b}_j\, e^{-2i\bar{k}_jx} \bar{N}_j(x)$$

For the inverse pb require: $C_j = b_j/a'(k_j)$, $\bar{C}_j = \bar{b}_j/\bar{a}'(\bar{k}_j)$

Scattering data that will be needed:

$$\mathcal{S}(k) = \{\rho(k), \{k_j, C_j\}, j = 1, ..., J;\ \bar{\rho}(k), \{\bar{k}_j, \bar{C}_j\}, j = 1, ..., \bar{J}\}$$

Symmetry Reductions

When $r(x) = \mp q^*(x)$:

$$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,k^*) \\ \mp N^{(1)}(x,k^*) \end{pmatrix}^* , \qquad \bar{M}(x,k) = \begin{pmatrix} \mp M^{(2)}(x,k^*) \\ M^{(1)}(x,k^*) \end{pmatrix}^*$$

$$\bar{a}(k) = a^*(k^*) , \qquad \bar{b}(k) = \mp b^*(k^*) ,$$

Thus the zeros of $a(k)$ and $\bar{a}(k)$ are paired, equal in number: $\bar{J} = J$

$$\bar{k}_j = k_j^* , \qquad \bar{b}_j = -b_j^* \qquad j = 1, \ldots, J$$

Only have evalues when $r(x) = -q^*(x)$: no evalues when $r(x) = +q^*(x)$

Symmetry Reductions–con't

For $r(x) = \mp q(x)$, $q(x) \in \mathbb{R}$:

$$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,-k) \\ \mp N^{(1)}(x,-k) \end{pmatrix}, \qquad \bar{M}(x,k) = \begin{pmatrix} \mp M^{(2)}(x,-k) \\ M^{(1)}(x,-k) \end{pmatrix}$$

$$\bar{a}(k) = a(-k), \qquad \bar{b}(k) = \mp b(-k),$$

Thus the zeros of $a(k)$ and $\bar{a}(k)$ are paired, equal in number: $\bar{J} = J$

$$\bar{k}_j = -k_j, \qquad \bar{b}_j = -b_j^* \qquad j = 1, \ldots, J$$

Only have evalues when $r(x) = -q(x) \in \mathbb{R}$: no evalues when $r(x) = +q(x)$

Since $r(x) = -q(x) \in \mathbb{R}$ satisfies $r(x) = -q(x)^*$ **both symmetry conditions hold**; so when k_j is an evalue so is $-k_j^*$; i.e. either the evalues come in pairs: $\{k_j, -k_j^*\}$ or they are pure Img

Symmetry Reductions–con't

For $r(x) = \mp q^*(-x)$

$$N(x,k) = \begin{pmatrix} \pm M^{(2)}(-x,-k^*) \\ M^{(1)}(-x,-k^*) \end{pmatrix}^{*}, \quad \bar{N}(x,k) = \begin{pmatrix} \pm \bar{M}^{(2)}(-x,-k^*) \\ \bar{M}^{(1)}(-x,-k^*) \end{pmatrix}^{*}$$

and the scattering data satisfies

$$a(k) = a^*(-k^*), \quad \bar{a}(k) = \bar{a}^*(-k^*), \quad \bar{b}(k) = \mp b^*(-k^*)$$

It follows that if $k_j = \xi_j + i\eta_j$ is a zero of $a(k)$ in UHP-k then $-k_j^* = -\xi_j + i\eta_j$ is also a zero of $a(k)$ in UHP-k etc

Also need data from 'right' which relate to data from 'left' – will not go into detail here

Inverse Problem

Recall: Soln process via IST:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{Direct Scattering}} & L: S(k,0) \\ & & \big\downarrow {\scriptstyle t:\ \text{time evolution: M}} \\ u(x,t) & \xleftarrow{\text{Inverse Scattering}} & S(k,t) \end{array}$$

— —

Operating with projection operators on the completeness relations after subtracting behavior at infinity and pole contributions

$$\begin{aligned} \frac{M(x,k)}{a(k)} &= \bar{N}(x,k) + \rho(k) e^{2ikx} N(x,k) \\ \frac{\bar{M}(x,k)}{\bar{a}(k)} &= N(x,k) + \bar{\rho}(k) e^{-2ikx} \bar{N}(x,k) \end{aligned}$$

yields integral eqs

Inverse Problem–Integral Eq

Genl $q(x), r(x)$:

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x}}{k-k_j} N_j(x) + \int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x} N(x,\zeta) d\zeta}{2\pi i(\zeta-(k-i0))}$$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2i\bar{k}_j x}}{k-\bar{k}_j} \bar{N}_j(x) - \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}(x,\zeta) d\zeta}{2\pi i(\zeta-(k+i0))}$$

where $N_j(x) = N(x,k_j)$, $\bar{N}_j(x) = \bar{N}(x,\bar{k}_j)$ We close the system by evaluating above eq at k_p and $\bar{k}_p$; $\quad p = 1, 2, ..., J$ resp.

By considering large k behavior from above eq and from direct Volterra integral eq we find reconstruction formulae for $r(x), q(x)$

Inverse Problem–Reconstruction Formulae

Genl $q(x), r(x)$:

$$r(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j N_j^{(2)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(\zeta) e^{2i\zeta x} N^{(2)}(x,\zeta) d\zeta$$

$$q(x) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j \bar{N}_j^{(1)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}^{(1)}(x,\zeta) d\zeta$$

Inverse Problem–With Symmetry

In each case can simplify prior integral eq with additional symmetry;
When $r(x) = \mp q^*(x)$ integral eq reduces to

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{j=1}^{J} \frac{\bar{C}_j e^{-2i\bar{k}_j x}}{k - \bar{k}_j} \bar{N}_j(x) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}(x,\zeta) d\zeta}{\zeta - (k + i0)}$$

with symmetry:

$$N(x,k) = \begin{pmatrix} N^{(1)}(x,k) \\ N^{(2)}(x,k) \end{pmatrix}, \quad \bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,k^*) \\ \mp N^{(1)}(x,k^*) \end{pmatrix}^*$$

$$\bar{\rho}(k) = \mp \rho(k)^* \quad k \in \mathbb{R}, \quad \bar{k}_j = k_j^*, \quad \bar{C}_j = \mp C_j^*$$

Note: system is closed by evaluating above integral eq at $k = k_p, \; p = 1, ..., J$

Inverse Problem–With Symmetry–con't

Recall:

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{J} \frac{\bar{C}_j e^{-2ik_j^* x}}{k - \bar{k}_j} \bar{N}_j(x) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}(x,\zeta) d\zeta}{\zeta - (k + i0)}$$

When $r(x) = \mp q(x) \in \mathbb{R}$ symmetry is:

$$N(x,k) = \begin{pmatrix} N^{(1)}(x,k) \\ N^{(2)}(x,k) \end{pmatrix}, \quad \bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,-k) \\ \mp N^{(1)}(x,-k) \end{pmatrix}$$

$$\bar{\rho}(k) = \mp \rho(-k) \;\; k \in \mathbb{R}, \quad \bar{k}_j = \{k_j^*, -k_j\}, \quad \bar{C}_j = \mp C_j$$

Inverse Problem–With Symmetry–con't

The case $r(x) = \mp q^*(-x)$ is somewhat more complex since we need efcns and completeness at **both** $\pm\infty$; in this case:

$$N(x,k) = \begin{pmatrix} \pm M^{(2)}(-x,-k^*) & ^* \\ M^{(1)}(-x,-k^*) & \end{pmatrix}^*, \quad \bar{N}(x,k) = \begin{pmatrix} \pm \bar{M}^{(2)}(-x,-k^*) \\ \bar{M}^{(1)}(-x,-k^*) \end{pmatrix}^*$$

Inverse Scattering–with Symmetry–con't

Use:

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{\bar{J}} \frac{\overline{C}_\ell \overline{N}(x,\bar{k}_\ell) e^{-2i\bar{k}_\ell x}}{k - \bar{k}_\ell} - \int_{-\infty}^{\infty} \frac{\bar{\rho}(\xi) e^{-2i\xi x} \overline{N}(x,\xi) d\xi}{2\pi i(\xi - (k + i0))}$$

Since $\overline{N}(x,k)$ is related to $\overline{M}^*(-x,k^*)$ also use

$$\overline{M}(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{B_\ell \; M(x,k_\ell) e^{-2ik_\ell x}}{k - k_\ell} + \int_{-\infty}^{\infty} \frac{R(\xi) e^{-2i\xi x} M(x,\xi) d\xi}{2\pi i(\xi - (k - i0))}$$

And since $M(x,k)$ is related to $N^*(-x,-k^*)$ this yields an integral eq for $N(x,k)$ (also have suitable symmetry for scatt data); Trace formula shows that only $b(k)$ and discrete data needed for inversion (add'l symmetries: $R(k) = \pm\rho^*(-k^*), B_\ell = \mp C_\ell^*$,...)

IST: Next Time Dependence

Soln process via IST:

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & L : S(k,0) \\
 & & \Big\downarrow \; t\text{: time evolution: M} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & S(k,t)
\end{array}
$$

IST: 2×2 Time Dependence

The associated M operator determines the evolution of the efcns
Taking into account BCs $\phi(x, k, t)$ satisfies

$$\partial_t \phi = \begin{pmatrix} A - A_\infty & B \\ C & -A - A_\infty \end{pmatrix} \phi \quad (E)$$

$$\text{where} \quad A_\infty = \lim_{|x| \to \infty} A(x, k)$$

Using completeness and evaluating $x \to \infty$:

$$\phi(x, k, t) = b(k, t)\psi(x, k, t) + a(k, t)\bar{\psi}(x, k, t) \sim \begin{pmatrix} a(t)e^{-ikx} \\ b(t)e^{ikx} \end{pmatrix}$$

Then as $x \to \infty$, (E) yields:

$$\begin{pmatrix} a_t e^{-ikx} \\ b_t e^{ikx} \end{pmatrix} = \begin{pmatrix} 0 \\ -2A_\infty b e^{ikx} \end{pmatrix}$$

IST: 2×2 Time Dependence–con't

Doing the same for $\bar{\phi}(x,k,t)$ find

$$\partial_t a = 0, \qquad \partial_t \bar{a} = 0$$
$$\partial_t b = -2A_\infty b, \qquad \partial_t \bar{b} = 2A_\infty \bar{b}$$

Thus then zero's of $a(k), \bar{a}(k)$ (evalues) $k_j, \bar{k}_j$ are const in time and for $\rho(k,t) = b(k,t)/a(k,t)$; $\bar{\rho} = \bar{b}(k,t)/\bar{a}(k,t)$:

$$\rho(k,t) = \rho(k,0)e^{-2A_\infty(k)t}, \qquad \bar{\rho}(k,t) = \bar{\rho}(k,0)e^{2A_\infty(k)t}$$

Similarly find:

$$C_j(t) = C_j(0)e^{-2A_\infty(k_j)t}, \qquad \bar{C}_j(t) = \bar{C}_j(0)e^{2A_\infty(\bar{k}_j)t}$$

Solitons–Reflectionless Potls

Can obtain pure soliton solutions; for genl $q(x,t), r(x,t)$ systems IST with: $\rho = 0, \bar{\rho} = 0$ i.e. reflectionless potls; inverse prob reduces to a linear algebraic system:

$$\bar{N}_l(x,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j(t) e^{2ik_j x} N_j(x,t)}{\bar{k}_l - k_j}$$

$$N_p(x,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{m=1}^{\bar{J}} \frac{\bar{C}_m(t) e^{-2i\bar{k}_m x} \bar{N}_m(x,t)}{k_p - \bar{k}_m},$$

with reconstruction:

$$r(x,t) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j(t) N_j^{(2)}(x,t)$$

$$q(x,t) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j(t) \bar{N}_j^{(1)}(x,t)$$

One Soliton Solns –With Symmetry

Using the time-dependence of $C_1(t)$ and symmetry:
$r(x,t) = -q(x,t)^*$
General one soliton soln:

$$q(x) = 2\eta e^{-2i\xi x + 2i\mathrm{Im}A_\infty(k_1)t - i\psi_0}\,\mathrm{sech}\left[2\left(\eta(x-x_0) + \mathrm{Re}A_\infty(k_1)t\right)\right]$$

where

$$k_1 = \xi + i\eta, \qquad C_1(0) = 2\eta e^{2\eta x_0 + i(\psi_0 + \pi/2)}$$

One Soliton Solns With Symmetry–con't

Special one soliton cases:

i) NLS: $r(x,t) = -q^*(x,t), k_1 = \xi + i\eta, A_\infty(k_1) = 2ik_1^2$

$$q(x,t) = 2\eta e^{-2i\xi x + 4i\left(\xi^2 - \eta^2\right)t - i\psi_0} \operatorname{sech}\left[2\eta\left(x - 4\xi t - x_0\right)\right]$$

ii) mKdV:

$r(x,t) = -q(x,t) \in \mathbb{R},\ \ k_1 = i\eta, \quad A_\infty(k_1) = -4ik_1^3 = -4\eta^3$

$$q(x,t) = 2\eta \operatorname{sech}\left[2\eta\left(x - 4\eta^2 t - x_0\right)\right]$$

iii) SG: $r(x,t) = -q(x,t) \in \mathbb{R},\ \ k_1 = i\eta, \quad A_\infty(k_1) = \frac{i}{4k_1} = \frac{1}{4\eta}$

$$q(x,t) = -\frac{u_x}{2} = -2\eta \operatorname{sech}\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right],$$

or in terms of u, a simple 'kink':

$$u(x,t) = 4\ \tan^{-1}\exp\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right]$$

One Soliton With Symmetry–con't

Nonlocal NLS: $r(x,t) = -q^*(-x,t)$: $\quad k_1 = i\eta, \bar{k}_1 = -i\bar{\eta}_1$

$$C_1(t) \;=\; C_1(0)e^{+4i\eta_1^2 t} \;=\; |c|e^{i(\varphi+\pi/2)}e^{+4i\eta_1^2 t}, \qquad |c| = \eta_1 + \bar{\eta}_1$$

$$\bar{C}_1(t) \;=\; \bar{C}_1(0)e^{-4i\bar{\eta}_1^2 t} \;=\; |\bar{c}|e^{i(\bar{\varphi}+\pi/2)}e^{-4i\bar{\eta}_1^2 t}, \qquad |\bar{c}| = \eta_1 + \bar{\eta}_1$$

Find a two parameter 'breathing' one soliton solution

$$q(x,t) = -\frac{2(\eta_1+\bar{\eta}_1)e^{i\bar{\varphi}}e^{-4i\bar{\eta}_1^2 t}e^{-2\bar{\eta}_1 x}}{1+e^{i(\varphi+\bar{\varphi})}e^{4i(\eta_1^2-\bar{\eta}_1^2)t}e^{-2(\eta_1+\bar{\eta}_1)x}}$$

Note $|c| = |\bar{c}| = \eta_1 + \bar{\eta}_1$ eigenvalues and 'norming' const related!

1-soliton reduces to NLS 1-soliton when $\eta_1 = \bar{\eta}_1$ and $\varphi + \bar{\varphi} = 0$

One Soliton With Symmetry–con't

Recall: two parameter 'breathing' one soliton solution

$$q(x,t) = -\frac{2(\eta_1+\overline{\eta}_1)e^{i\overline{\varphi}}e^{-4i\overline{\eta}_1^2 t}e^{-2\overline{\eta}_1 x}}{1+e^{i(\varphi+\overline{\varphi})}e^{4i(\eta_1^2-\overline{\eta}_1^2)t}e^{-2(\eta_1+\overline{\eta}_1)x}}$$

Note that there are singularities at $x=0$ with:

$$1+e^{i(\varphi+\overline{\varphi})}e^{4i(\eta_1^2-\overline{\eta}_1^2)t}=0 \quad \text{or at}$$

$$t = t_n = \frac{(2n+1)\pi-(\varphi+\overline{\varphi})}{4(\eta_1^2-\overline{\eta}_1^2)}\ , \qquad n\in\mathbb{Z}$$

Singularity disappears when $\eta_1=\overline{\eta}_1$ and $\varphi+\overline{\varphi}\neq(2n+1)\pi, n=\mathbb{Z}$

Conserved quantities

$a(k,t)$ is conserved in time; it can be related to the conserved quantities. This follows from the relation

$$a(k,t) = \lim_{x\to+\infty} \phi^{(1)}(x,k;t)e^{ikx}$$

and the large k asymptotic expn for the efcn: $\phi = \left(\phi^{(1)}, \phi^{(2)}\right)^T$
The first few conserved quantities are:

$$\begin{aligned} C_1 &= -\int q(x)r(x)dx, \qquad C_2 = -\int q(x)r_x(x)dx \\ C_3 &= \int \left(q_x(x)r_x(x) + (q(x)r(x))^2\right) dx \end{aligned}$$

Similar ideas lead to conservation laws

Conserved quantities–con't

For example, with the reductions $r = \mp q^*$ these constants of the motion can be written as

$$\begin{aligned} C_1 &= \pm \int |q(x)|^2 \, dx, \qquad C_2 = \pm \int q(x) q_x^*(x) dx \\ C_3 &= \int \left(\mp |q_x(x)|^2 + |q(x)|^4 \right) dx \end{aligned}$$

Inverse Pb–Triangular Representations: Towards GLM

For general $q(x), r(x)$:

Assuming triangular representations for $N, \bar{N}$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^{+\infty} K(x,s) e^{ik(s-x)} ds, \quad s > x, \quad \mathrm{Im} k \geq 0$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^{+\infty} \bar{K}(x,s) e^{-ik(s-x)} ds, \quad s > x, \quad \mathrm{Im} k \leq 0$$

substituting into prior integral eq and taking FTs, GLM eq follow

Inverse Problem–via GLM Eq–con't

For general $q(x), r(x)$ find

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{+\infty} K(x,s)F(s+y)ds = 0$$

$$K(x,y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) + \int_x^{+\infty} \bar{K}(x,s)\bar{F}(s+y)ds = 0$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \rho(\xi)e^{i\xi x}d\xi - i\sum_{j=1}^{J} C_j e^{ik_j x}$$

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\rho}(\xi)e^{-i\xi x}d\xi + i\sum_{j=1}^{\bar{J}} \bar{C}_j e^{-i\bar{k}_j x}$$

GLM: Reconstruction – Symmetry

Reconstruction for general $q(x), r(x)$

$$q(x) = -2K^{(1)}(x,x), \qquad r(x) = -2\bar{K}^{(2)}(x,x)$$

Symmetry reduces the GLM eq; with $r(x) = \mp q(x)^*$ find

$$\bar{F}(x) = \mp F^*(x), \quad \bar{K}(x,y) = \begin{pmatrix} K^{(2)}(x,y) \\ \mp K^{(1)}(x,y) \end{pmatrix}^*$$

In this case the GLM eq reduces to

$$K^{(1)}(x,y) = \pm F^*(x+y) \mp \int_x^{+\infty} ds \int_x^{+\infty} ds' K^{(1)}(x,s')F(s+s')F^*(y+s)$$

for $y > x$; When $r(x) = \mp q(x) \in \mathbb{R}$ then $F(x)$ and $K(x,y)$ are $\in \mathbb{R}$

Conclusion and Remarks

- Discussed: in these lectures:
- Compatible linear systems–Lax Pairs–2×2 systems
- IST method–nonlinear Fourier transform
- IST associated with KdV
- IST for general q, r: 2×2 systems
- q, r systems with symmetry:
    - $r(x,t) = \mp q^*(x,t)$: NLS
    - $r(x,t) = \mp q(x,t) \in \mathbb{R}$; mKdV, SG
    - $r(x,t) = \mp q^*(-x,t)$: nonlocal NLS
- Not discussed– long time asymptotic analysis where solitons and similarity solns/Painleve fcns (e.g. for KdV/mKdV) play important roles

Conclusion and Remarks

- May also carry out IST for many other systems, some physically interesting
  - Higher order and more complex $1 + 1$d PDE evolution systems: N Wave eq; Boussinesq eq
  - Nonlocal eq such as Benjamin-Ono (BO) and Intermediate Long wave eq
  - Discrete problems: e.g. Toda lattice, discrete ladder systems, integrable discrete NLS
  - $2 + 1$d systems such as Kadomtsev-Petviashvili (KP), Davey-Stewartson, N Wave systems
  - In $2 + 1$ there are some important extensions/new ideas needed for IST: notably DBAR problems: e.g. KPII

References

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